REDUCING THE STRESS CONCENTRATION AROUND CIRCULAR HOLES IN LAMINATED COMPOSITES

Dr. Rosen T. Tenchev

Dept. Strength of Materials, Technical University - Sofia, Bulgaria

Visiting researcher at:

Mechanics Division, Dept. of Mathematics, University of Oslo

Prof. Magne K. Nygard

Mechanics Division, Dept. of Mathematics, University of Oslo

Dr. Andreas Echtermeyer

Dept. for Hydrodynamics, Structures and Materials

Det Norske Veritas Research AS

SUMMARY

The stress concentration around a hole in a laminated composite may be decreased by increasing the thickness (bonding of 2D panels) in this area. The problem may be described as an infinite orthotropic membrane with a finite circular orthotropic inclusion (ring) and a circular hole in the middle of the ring under arbitrary in-plane loading. A method for the calculation of the stress concentration at the edge of the hole is presented. The method is based on the analytical solution for the case of infinite inclusion. The finite dimensions of the inclusion are taken into account by means of a correction coefficient. It is established empirically by processing a large amount of finite element results.

Key words: composite laminates; stress concentration; finite element analysis

NOTATIONS

Scalars:

E - Young's modulus

G - Shear modulus

ν - Poisson's ratio (when loading is along x $\varepsilon_y=-\nu_{xy}\varepsilon_x$)

t - Thickness

r - Radius of the hole in the laminate

R - Outside radius of the reinforcing ring

x,y - Principal material axes for orthotropic laminate

φ - Angle between x and the direction of the remote loading σ^∞

θ - Angular coordinate of a point at the hole edge.
Also: The direction tangent to the hole edge at that point.

ρ - Direction perpendicular to θ at the same point.

σ - A component of the stress vector $\{\sigma\}$

K_t - Stress concentration factor

s - Ratio of the applied remote load, $s = \sigma_y^\infty / \sigma_x^\infty$

Y - Ratio between the stress concentration factors of a laminate with finite and infinite dimensions of the reinforcing ring.

X - Ratio between the stiffness of the laminate and the reinforcing ring.

C - Coeff. for computing the remote stress in the laminate and reinforcing ring from the averaged one.

ξ - Coeff. accounting for the mismatch of the Poisson's ratioes of the laminate and the reinforcing ring.

Subscripts:

x,y - reference to the x,y principal material axes

θ,ρ - reference to the θ,ρ directions (axes)

L - reference to the original laminate

R - reference to the reinforcing ring

A - reference to the assembly: laminate + reinforcing ring

, (comma) - separates the subscripts, not a differentiation symbol

Superscripts:

∞ - reference infinite dimensions or remote stress

FEM - reference to Finite Element Method results

E - reference to Estimated values

Vectors:

$\{\sigma\}$ - Stress vector

$\{\varepsilon\}$ - Strain vector

Matrices:

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$ - Material stiffness matrix used in: $\{\sigma\}=[Q]\{\varepsilon\}$

$$Q_{11} = \frac{E_x}{1 - \nu_{xy}\nu_{yx}} \qquad Q_{22} = \frac{E_y}{1 - \nu_{xy}\nu_{yx}}$$

$$Q_{12} = \frac{\nu_{yx}E_x}{1 - \nu_{xy}\nu_{yx}} = Q_{21} = \frac{\nu_{xy}E_y}{1 - \nu_{xy}\nu_{yx}}$$

$$Q_{66} = G_{xy}$$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$$ - Compliance matrix used in: $\{\varepsilon\}=[S]\{\sigma\}$

$$S_{11} = \frac{1}{E_x} \qquad S_{22} = \frac{1}{E_y}$$

$$S_{12} = -\frac{\nu_{yx}}{E_y} = S_{21} = -\frac{\nu_{xy}}{E_x}$$

$$S_{66} = \frac{1}{G_{xy}}$$

Examples:

$E_x, E_y, E_\theta, E_\rho$ - Young's modulus for an orthotropic laminate along x,y,θ,ρ axes

E_R - Young's modulus for the isotropic reinforcing ring

1. Description of the problem

Consider a 2D laminated composite used, for example, as a bulkhead in a ship. Let us suppose that the bulkhead's design has been correct and that the stresses are everywhere within the allowable limits. As an afterthought a hole has to be drilled (to pass some communication lines, pipes, etc.), Figure 1, which has not been considered during the design. There would be a stress concentration, Figure 5a, around the hole and dangerous stresses may occur which may cause failure of the laminate.

The aim is to decrease the stress concentration.

The easiest solution is to increase the thickness of the laminate around the hole, by means of bonding 2D panels (which we shall call a reinforcing ring) symmetrically on both sides of the laminate, Figures 1b and 1c.

The problem is to choose the right type of reinforcing ring (i.e. material properties and thickness) which can be solved if we know *how to compute the stress concentration for the new assembly - original laminate plus reinforcing ring.* If the reinforcing ring has the same dimensions as the original laminate, Figure 1b, the assembly may be treated as a new laminate with new orthotropic material properties and an analytical model[1] can be used. Since an infinite laminate is considered for the theoretical solution this reinforcing ring will be infinite, too.

In practice, however, the reinforcing ring will have finite dimensions, Figure 1c. The problem may be described as an orthotropic membrane (infinite or finite dimensions) with a stiffer inclusion of finite dimensions and a circular hole. The authors do not know of a theoretical solution for this case and they doubt that such one may exist in a closed form.

In this paper it is proposed to use the analytical expression for the stress concentration[1] applied to an infinite assembly (original laminate and reinforcing ring) plus an empirically derived correction factor for the finite dimensions of the reinforcing ring.

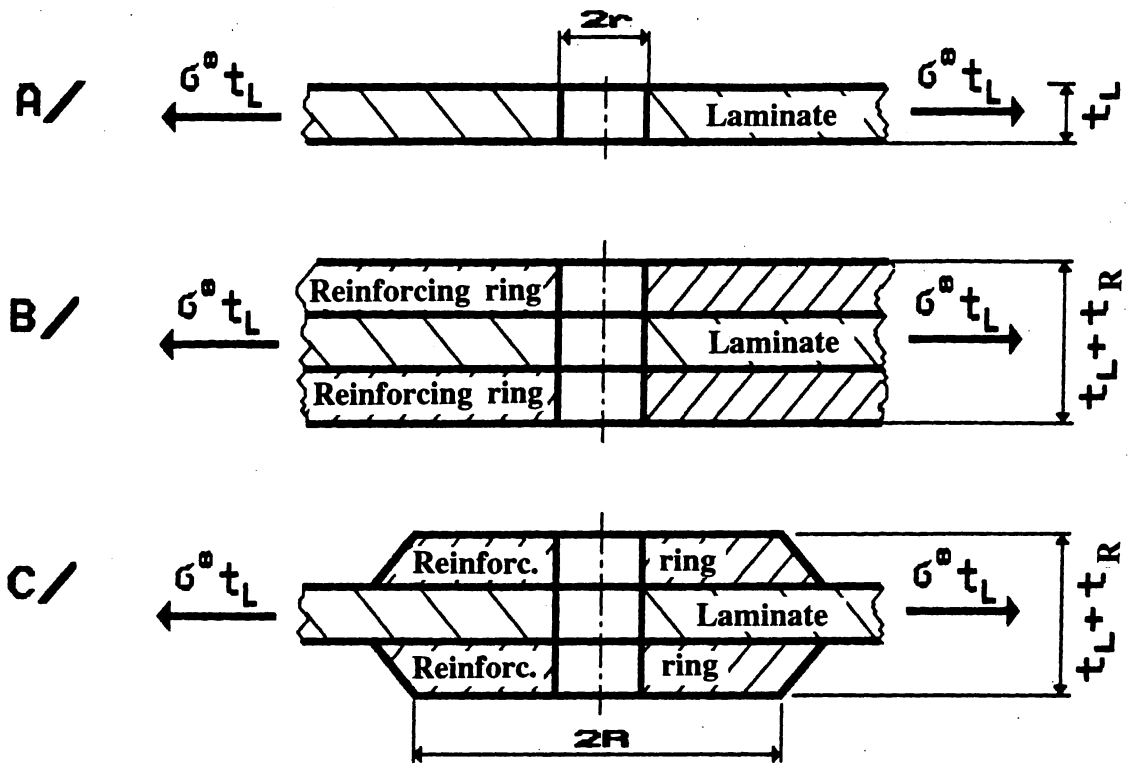

Figure 1.

A/ Infinite laminate with a hole - high stress concentration

B/ Infinite laminate with infinite reinforcing ring - analytical solution

C/ Infinite laminate with finite reinforcing ring - no analytical solution

2. Theoretical Analysis

2.1. The analytical stress concentration for an orthotropic plate

The problem will be simplified by the following assumptions:

A1) The laminated composite can be modeled as a 2D plane stress, orthotropic problem: symmetric lay-up; in-plane loading.

A2) The hole is small compared to the dimensions of the laminate, i.e. a hole in an infinite plate.

The analytical solution given by Lekhnitskii[1] for the stress σ_θ at the edge of a circular hole in an infinite orthotropic plate (membrane), Figure 2, is:

$$\sigma_\theta = \sigma^\infty \left(\frac{E_\theta}{E_x}\right) \left\{ \left[-\cos^2\varphi + (k+n)\sin^2\varphi \right] k \cos^2\theta + \right.$$
$$+ \left[(1+n)\cos^2\varphi - k\sin^2\varphi \right] \sin^2\theta -$$
$$\left. - n(1+k+n)\sin\varphi\cos\varphi\sin\theta\cos\theta \right\} \tag{1}$$

Where:

$$k = \sqrt{\frac{E_x}{E_y}} \tag{2}$$

$$n = \sqrt{2\left(k - \nu_{xy}\right) + \frac{E_x}{G_{xy}}} \tag{3}$$

$$\frac{1}{E_\theta} = \frac{\sin^4\theta}{E_x} + \left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right) \cos^2\theta \sin^2\theta + \frac{\cos^4\theta}{E_y} \tag{4}$$

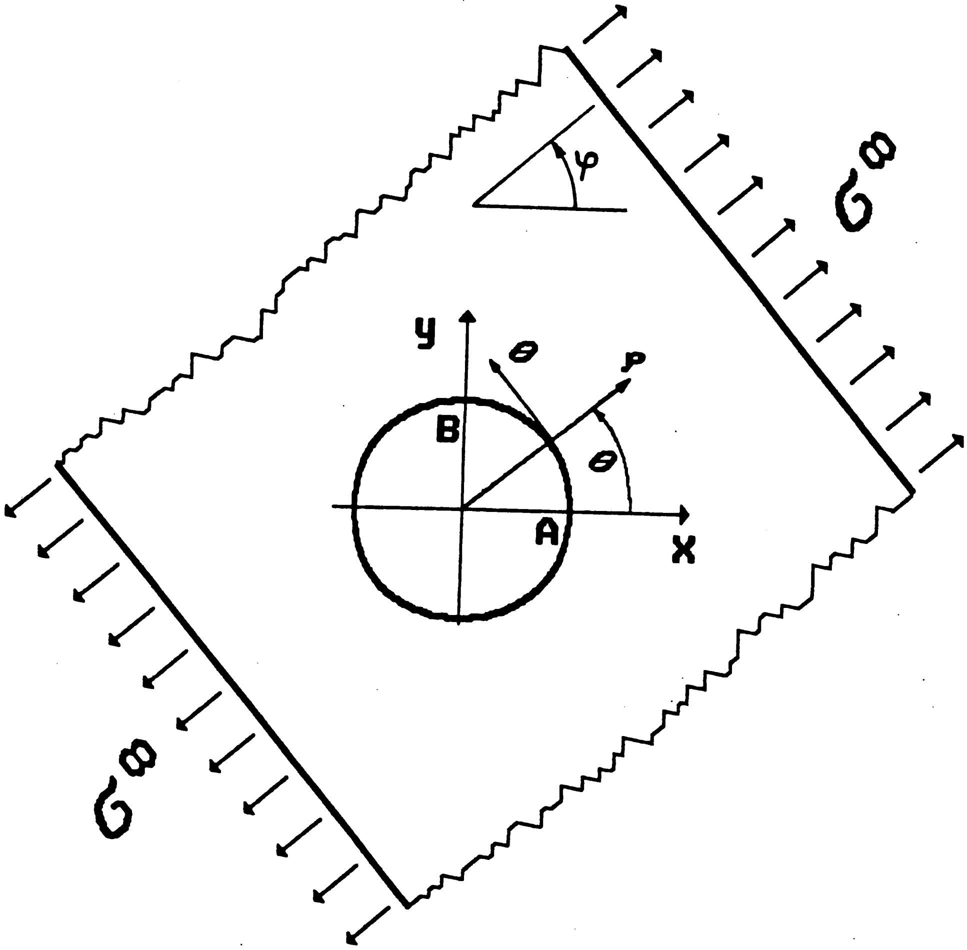

Figure 2.
A circular hole in a infinite laminate
General unidirectional loading

2.2. Extreme values for the stress concentration factor

Inspecting eqns (1) to (4) with a realistic values of the material constants it can be seen that the extreme values of σ_θ occur when the load σ^∞ is applied in the direction of the principal material axis:

Loading along x: $\varphi = 0$ and from eqn (1):

$$\sigma_\theta = \sigma_x^\infty \left(\frac{E_\theta}{E_x}\right)\left[\left(-k\right)\cos^2\theta + \left(1+n\right)\sin^2\theta\right] \qquad (5)$$

At point A, $\theta = 0$:

$$\sigma_\theta = -\frac{1}{k}\,\sigma_x^\infty \qquad (6)$$

At point B, $\theta = 90^\circ$:

$$\sigma_\theta = (1+n)\,\sigma_x^\infty \qquad (7)$$

Loading along y: $\varphi = 90^\circ$ and from eqn (1)

$$\sigma_\theta = \sigma_y^\infty \left(\frac{E_\theta}{E_x}\right)\left[\left(n+k\right)\cos^2\theta - \sin^2\theta\right]k \qquad (8)$$

At point A, $\theta = 0$:

$$\sigma_\theta = (n+k)\,\sigma_y^\infty \qquad (9)$$

At point B, $\theta = 90^\circ$:

$$\sigma_\theta = -k\,\sigma_y^\infty \qquad (10)$$

In the case of general loading there are several failure criteria but if we accept the simplest one that the components of the remote stress must not excede the maximum allowed unidirectional stress (for the unnotched laminate) then we have to consider the principal stresses. So further on we shall assume that σ_x^∞ and σ_y^∞ are principal stresses.

Let the biaxial remote stress field be:

$$\sigma_x^\infty = \sigma^\infty \qquad \sigma_y^\infty = s\sigma^\infty \qquad \tau_{xy}^\infty = 0 \tag{11}$$

where s defines the difference in the loading in both directions.

Using the principal of superposition the stress concentration is:

At point A, $\theta = 0$:

$$\sigma_\theta = \left(-\frac{1}{k} + s(n+k) \right) \sigma^\infty \tag{12}$$

At point B, $\theta = 90^o$:

$$\sigma_\theta = \left(1 + n - sk \right) \sigma^\infty \tag{13}$$

When $E_x > E_y$ then $k > 1$. Usually the stiffer direction can carry more load and $-1 \leq s \leq 1$. The highest values of σ_θ will be achieved when $s = -1$, i.e. biaxial tension-compression (eqn (11), Figure 3) The maximum absolute value of σ_θ is at point B, direction - along x, and we have max $\sigma_\theta \equiv \sigma_{\theta=90} \equiv \sigma_{\theta,B} \equiv \sigma_{x,B}$.

The maximum stress concentration factor in the case of biaxial tension - compresion is:

$$\frac{\sigma_{x,B}}{\sigma^\infty} = K_t^\infty = (1 + n + k) \tag{14}$$

To allow for arbitrary loading, which is still safe for the unnotched laminate, we shall assume:

A3) The hole is drilled in the area with the worst stress combination which will cause the highest stress concentration given by eqn (14).

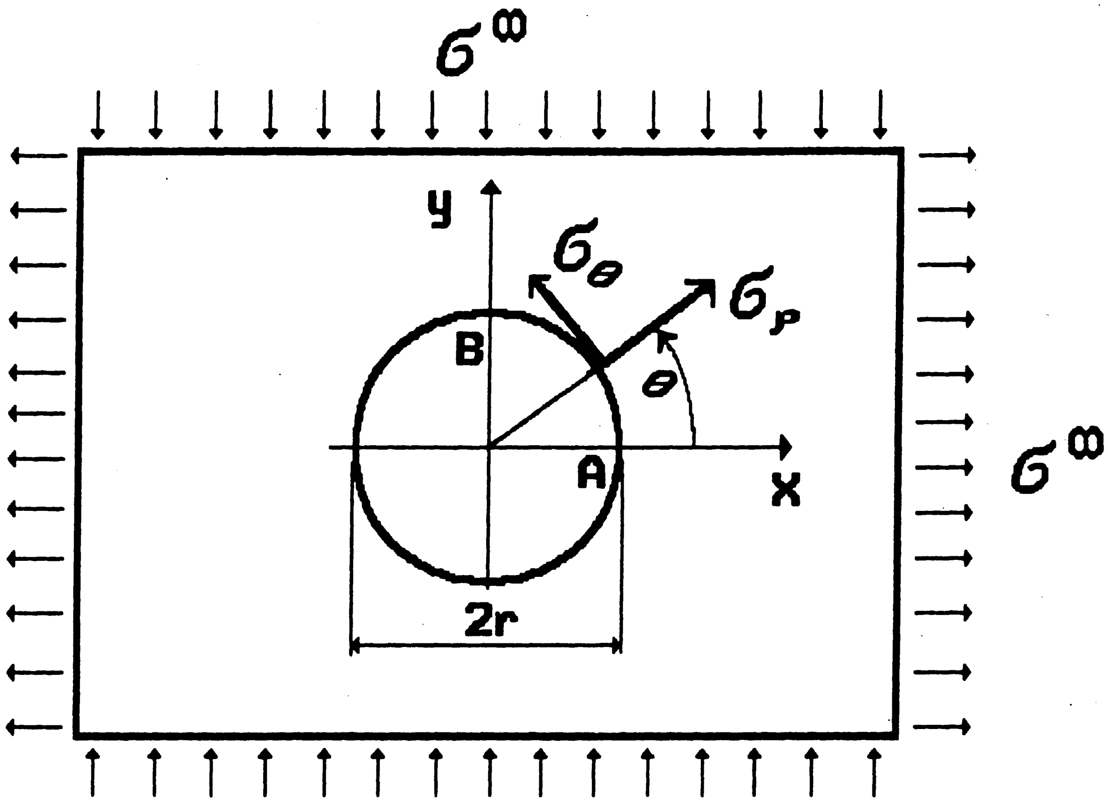

Figure 3.

The 2D tension-compression loading along the principal material axes that gives the maximum stress concentration:

At point A when $E_y > E_x$; At point B when $E_x > E_y$

2.3. The case of infinite dimensions of the reinforcing ring

Further assumptions:

A4) The reinforcing ring is bonded firmly and continuously on both sides of the original laminate. The resultant assembly remains symmetric about the mid-surface of the original laminate and in-plane loading will not produce an out-of-plane deformation.

A5) Isotropic reinforcing ring. When the procedures outlined here are used for repairs it may not be possible to determine the orthotropic axis of the original laminate. To preserve orthotropic properties for the resultant assembly the reinforcing ring must be isotropic.

The strains in the assembly, laminate and reinforcing ring are equal. The stress-strain relationship is, respectively:

$$\begin{aligned} \{\sigma\}_A &= [Q]_A\{\varepsilon\} \\ \{\sigma\}_L &= [Q]_L\{\varepsilon\} \\ \{\sigma\}_R &= [Q]_R\{\varepsilon\} \end{aligned} \qquad (15)$$

where $\{\sigma\} = \{\sigma_x \;\; \sigma_y \;\; \tau_{xy}\}$ and $\{\varepsilon\} = \{\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy}\}$.

The averaged stress in the assembly whitout a hole $\{\sigma\}_A$, Figure 4, can be computed from:

$$\{\sigma\}_A(t_L+t_R) = \{\sigma\}_L t_L + \{\sigma\}_R t_R = \{\sigma\}^{\infty} t_L \qquad (16)$$

The material stiffness matrix of the assembly, from eqns (15) and (16) is:

$$[Q]_A = \frac{[Q]_L t_L + [Q]_R t_R}{t_L + t_R} \qquad (17)$$

The compliance matrix for the assembly is:

$$[S]_A = [Q]_A^{-1} \qquad (18)$$

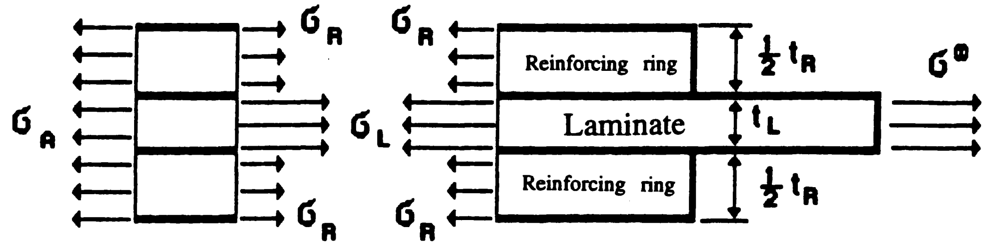

Figure 4.
The distribution of the membrane stress
through the thickness of the assembly - laminate and reinforcing ring

The material properties of the assembly can be recovered from the terms in the compliance matrix:

$$E_{x,A} = \frac{1}{S_{11,A}} \qquad E_{y,A} = \frac{1}{S_{22,A}}$$
$$G_{xy,A} = \frac{1}{S_{66,A}} \qquad \nu_{xy,A} = - E_{x,A} S_{12,A} \tag{19}$$

The stress $\sigma_{\theta,A}$ for the assembly can be calculated from eqns (1) to (4) using the material properties given by eqn (19).

2.4. Distribution of $\sigma_{\theta,A}$ in the laminate and the reinforcing ring

The stress in the laminate and the reinforcing ring, from eqns (15)-(18), is:

$$\{\sigma\}_L = [Q]_L[S]_A\{\sigma\}_A$$
$$\{\sigma\}_R = [Q]_R[S]_A\{\sigma\}_A \tag{20}$$

If the θ, ρ coordinate system is considered (Figure 2), the stress components at the edge of the hole (an unloaded boundary) are:

$$\sigma_{\theta,A} \neq 0 \qquad \sigma_{\rho,A} = 0 \qquad \tau_{\theta\rho} = 0 \tag{21}$$

From eqns (16),(17), (18) and (20) we obtain:

$$\sigma_{\theta,L} = C_L \, \sigma_\theta^\infty \qquad \sigma_{\theta,R} = C_R \, \sigma_\theta^\infty \tag{22}$$

where:

$$C_L = \frac{E_{\theta,L} t_L}{E_{\theta,L} t_L + \xi E_R t_R}$$
$$C_R = \frac{E_R \, t_L}{\frac{1}{\xi} E_{\theta,L} t_L + E_R t_R} \tag{23}$$

and:

$$\xi = \frac{E_{\rho,L}t_L\left(1 - \nu_{\theta\rho,L}\nu_R\right) + E_R t_R\left(1 - \nu_{\theta\rho,L}\nu_{\rho\theta,L}\right)}{E_{\rho,L}t_L\left(1 - \nu_R\,\nu_R\right) + E_R t_R\left(1 - \nu_R\,\nu_{\rho\theta,L}\right)} \qquad (24)$$

E_ρ can be computed from eqn 4, using an angle $\theta+90^\circ$ instead of θ.

The coefficient ξ being different from unity is a consequence of the mismatch of the Poisson's ratios $\nu_{\theta\rho,L}$, ν_R of the lamninate and the reinforcing ring..

The stress concentration factor for the assembly $K_{t,A}$ at any point for any loading can be calculated from eqn (1). Its maximum value (on the diameter perpendicular to the stiffer material direction; biaxial tension-compression) is given by eqn (14). The material properties of the assembly are given by eqn (19).

The stress concentration factors in the laminate and in the reinforcing ring, when they are with infinite dimensions, are:

$$K^{\infty}_{t,L} = C_L\, K^{\infty}_{t,A} \qquad\qquad K^{\infty}_{t,R} = C_R\, K^{\infty}_{t,A} \qquad (25)$$

2.5. Selection of a finite reinforcing ring

2.5.1 Shape

A circular shape of the reinforcing ring is recommended. The worst loading is when the principal stress directions coincide with the principal material directions and these directions may be unknown. To allow for an arbitrary orientation, in addition to being isotropic, the reinforcing ring must be symmetric about any axis, i.e. to have a circular shape.

2.5.2. Radius of the ring

The stress concentration is localized in a small area around the hole, therefore the reinforcing ring does not have to be infinite. The stress distribution along the line passing through the center of the hole and perpendicular to the remote loading σ_x^∞ is[1]:

$$\sigma_x(0,y) = \frac{\sigma_x^\infty}{2}\left\{2 + \left(\frac{r}{y}\right)^2 + 3\left(\frac{r}{y}\right)^4 - \left(K_t-3\right)\left[5\left(\frac{r}{y}\right)^6 - 7\left(\frac{r}{y}\right)^8\right]\right\} \qquad (26)$$

for $y \geq r$, where r is the hole radius.

A reinforcing ring reduces the stress concentration at point B as shown in Figure 5b. The inside radius of the reinforcing ring is equal to the hole radius r in the laminate, the outside radius of the reinforcing ring R may be computed from eqn (26) so as to have an acceptable value for the stress concentration.

For isotropic materials $K_t^\infty = 3$, for unidirectional carbon-fiber epoxy laminate - $K_t^\infty \approx 6.5$, for laminates dominated by the ±45° plies - $K_t^\infty \approx 2.5$.

Some values of the stress concentration $\sigma_x(y)/\sigma_x^\infty$ are given in Table 1.

	y=1.5r	y=4.0r
K_t^∞=2.5	1.5600	1.0374
K_t^∞=3.0	1.5185	1.0371
K_t^∞=6.5	1.2283	1.0352

Table 1

Values of $\sigma_x(y)/\sigma_x^\infty$ according to eqn. 26

The further discussions will consider R values between 1.5r and 4r. For values of R<1.5r the stress concentration outside the reinforcing is too high. For R>4r it is negligible.

For infinite laminate the stress at B^{∞}, Figure 5b, is equal to the applied remote stress. For laminates with finite width, $R/W \neq 0$, an approximate estimate of the remote stress at B^{∞} can be done if we assume that the stress is reduced to zero in the reinforced area and σ_x on the line B'-B^{∞} is constant. From static equilibrium:

$$\text{at } B^{\infty}: \qquad \sigma_x = \left(\frac{1}{1 - R/W}\right) \sigma_x^{\infty} \tag{27}$$

3. Finite Element Analyses

Finite element analyses will be used to find the true stress concentration factor in the laminate and in the reinforcing ring when the latter is of finite dimensions. The loading will be the biaxial tension-compression one, parallel to the principal material axes of the laminate, giving the highest stress concentration factor.

A correction factor Y^E will be established. It will describe the difference in the stress concentration factors in a laminate with infinite and finite dimension of the reinforcing ring. It is expected to be a function of the outside radius of the reinforcing ring, the material properties and the thickness of the laminate and the reinforcing ring.

The estimated stress concentration factors $K^E_{t,\perp}$ and $K^E_{t,R}$ for a problem with *finite dimensions* of the reinforcement ring is to be approximately calculated as:

$$K^E_{t,\perp} = Y^E K^{\infty}_{t,\perp}$$
$$K^E_{t,R} = Y^E K^{\infty}_{t,R} \tag{28}$$

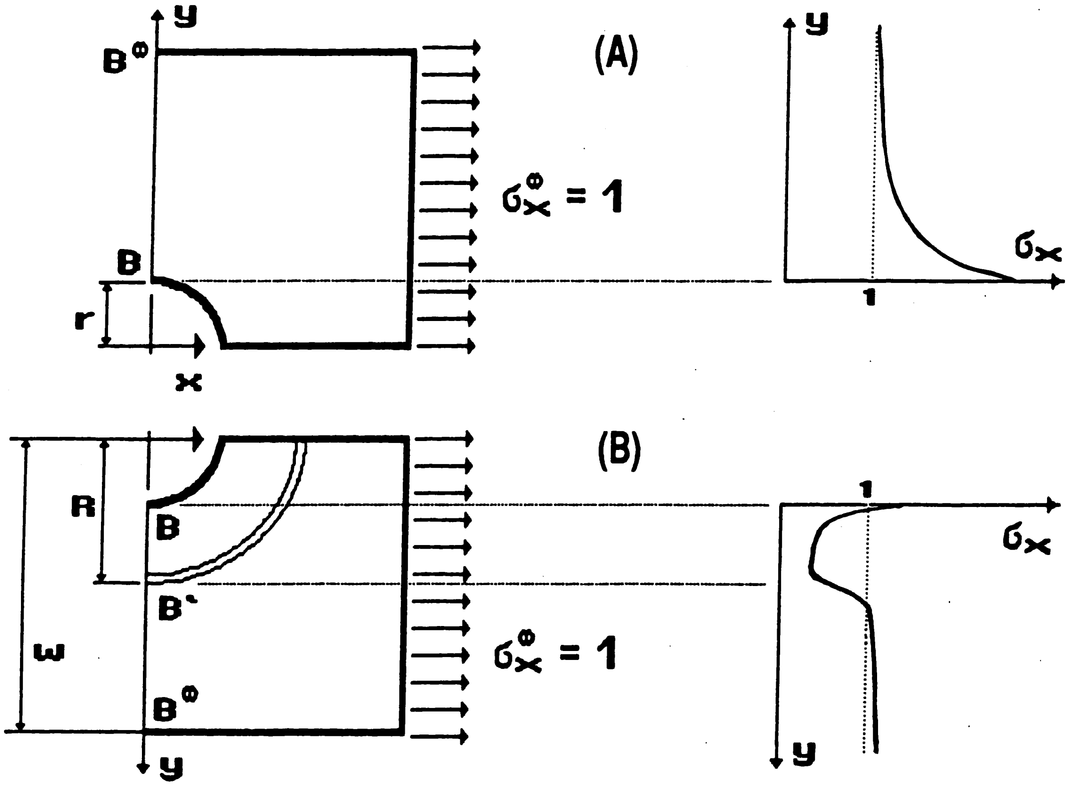

Figure 5.

The stress distribution along line B-B^{∞}

(A) without reinforcing ring (B) with reinforcing ring where $K^{\infty}_{t,L}$ and $K^{\infty}_{t,R}$ are the analytical stress concentration factors for the same problem, but with *infinite dimensions* of the reinforcing ring, given by eqn (25), and the subscripts L and R stand for the laminate and the reinforcing ring, respectively.

It is assumed that Y^E is the same for the laminate and the reinforcing ring. The numerical results, presented later, show that this assumption is very accurate.

3.1. FE model description

The stresses in an infinite laminate with a hole will be solved by FEM as a square laminate (semi-width W) with a small hole (W/r = 16, r=1mm). Due to symmetry one quarter will be analyzed. An illustrative model, out of scale, is shown on Figure 6. Four examples with R/r = 1.5; 2; 3 and 4 are meshed. The mesh density has been chosen in a way that the FE results and the analytical calculations for infinite reinforcing ring given by eqn (25) differ by less than 3%.

A mesh for the model R/r=4 is shown on Figure 7. A reinforcing ring is applied by repeating an element definition with the reinforcing ring material properties and thickness. For example, the model on Figure 7 when there is no reinforcing ring has 130 elements, with infinite reinforcing ring - 260 elements, with finite reinforcing ring of R=4r - 200 elements and 437 nodes for all cases. Plane stress, isoparametric, 8-node elements are used.

The analyses are performed with a FEM software developed by Dr. Tenchev and the standard FEM procedures are included in nested do-loops which provide for the required variations described in section 3.2. The stress σ_x at point B (in the laminate and in the reinforcing ring), which for a unit remote stress is numerically equal the stress concentration factor, is of interest.

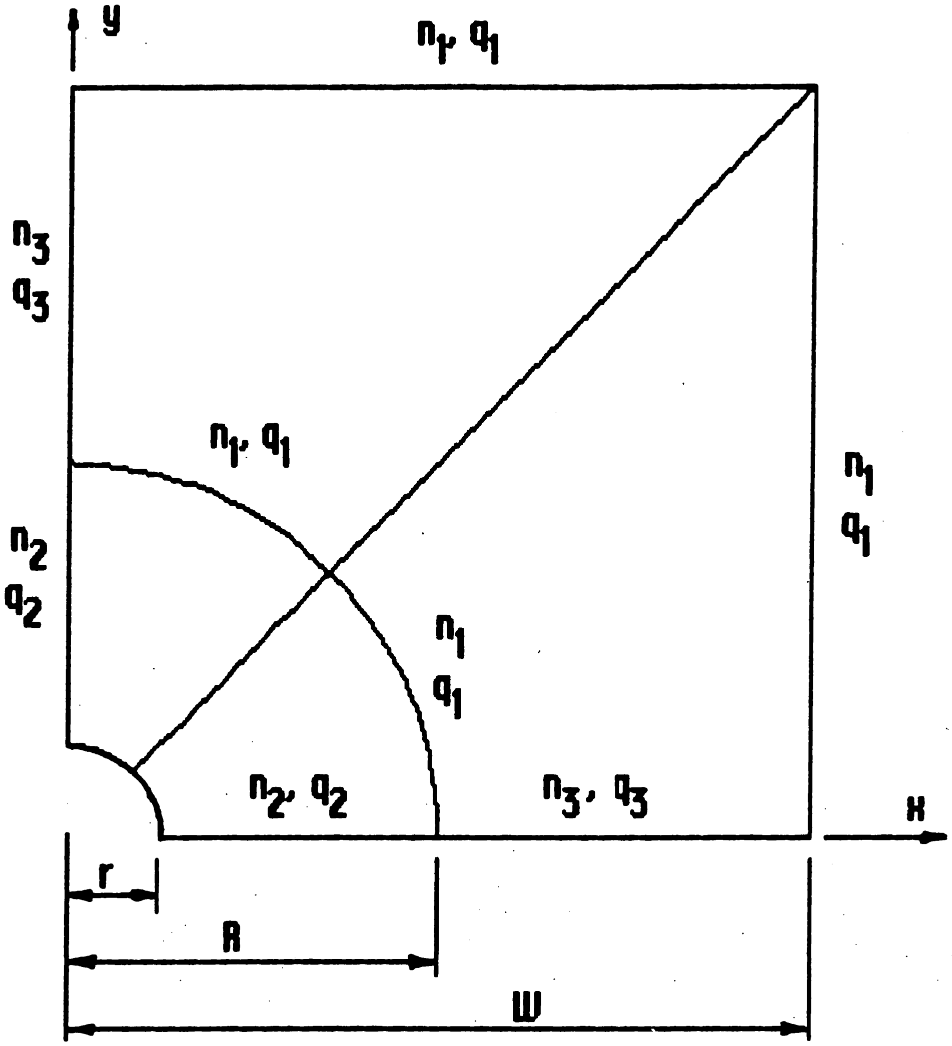

Figure 6.
Geometry and mesh density parameters of the FEM model

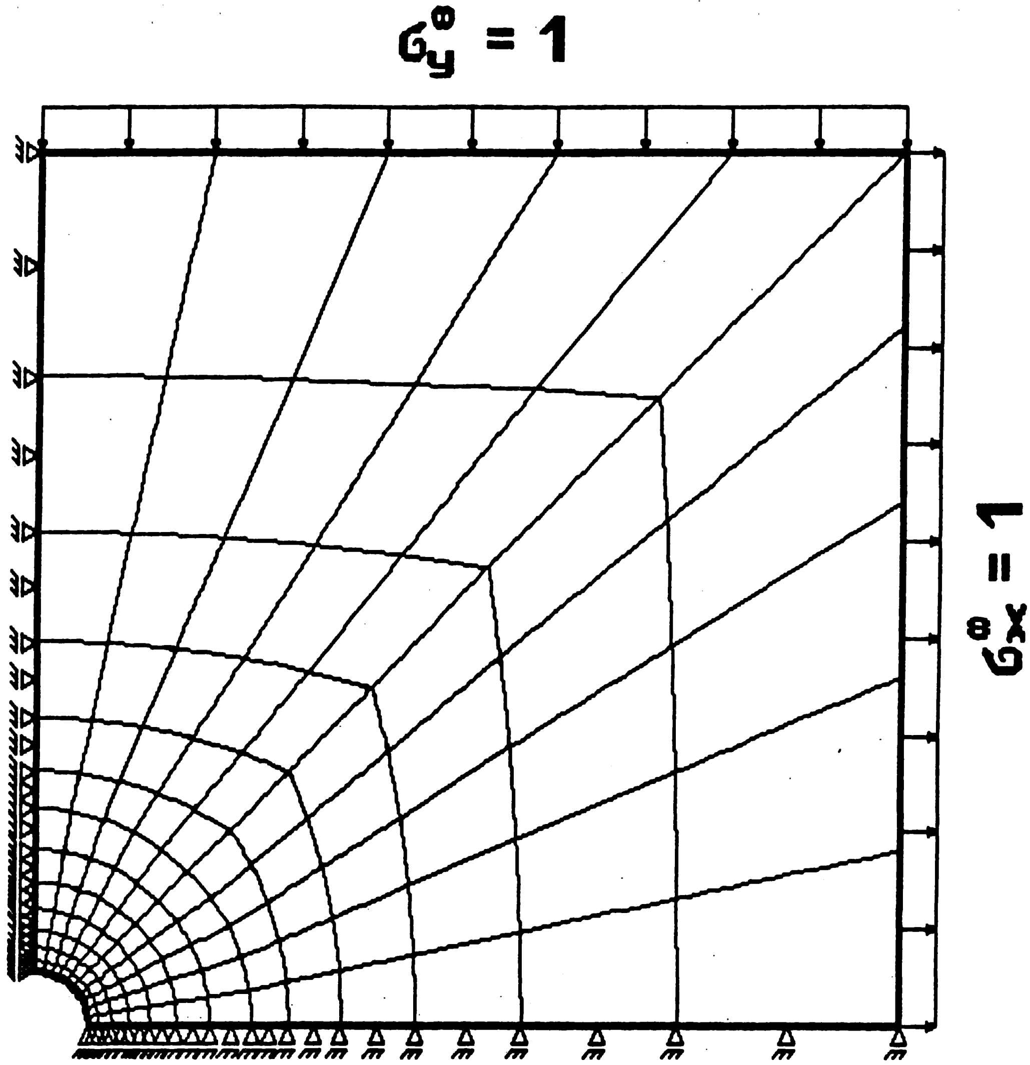

Figure 7.
A typical FEM mesh, R/r = 4, d.o.f. = 874

The number of elements n_i and the factor q_i on the edges of the model, Figure 6, are given in Table 2. The q_i is a factor of a geometric progression by which the consecutive elements are getting smaller towards the hole.

R/r	n_1	q_1	n_2	q_2	n_3	q_3
1.5	5	1.0	4	0.70	9	0.90
2.0	5	1.0	5	0.70	8	0.85
3.0	5	1.0	6	0.70	7	0.85
4.0	5	1.0	7	0.70	6	0.80

Table 2
Mesh density parameters

The principal material axes are parallel to the loading and the sides of the model.

3.2. Variation of the material properties and thickness

Two typical types of composite material are considered with the following material properties for unidirectional, (100% 0^o plies) laminate[2]. Direction 11 - along the fibers, 22 - transverse to fibers.

Glass -fiber/epoxy:

E_{11} = 42 GPa; E_{22} = 14 GPa; G_{12} = 7 GPa; ν_{12} = 0.25

Carbon-fiber/epoxy (T300):

E_{11} = 156 GPa; E_{22} = 13 GPa; G_{12} = 7 GPa; ν_{12} = 0.23

To have varying material properties of the laminate the FEM data has been generated as follow:

It is supposed that the laminate is made of 0^o, 90^o and $\pm 45^o$ plies. The fraction of 90^o plies, P_{90}, has been varied from 0% to 100% in steps of 10%. The total amount of $\pm 45^o$ plies, P_{45}, has been varied from 0% to 100% in steps of 10%, too. The amount of the 0^o plies is $P_{00} = 1 - P_{90} - P_{45}$ and it must not be negative. The material properties of these laminates are computed in a standard way, given in Appendix A. Only those laminates were analyzed which have $E_x \geq E_y$, and for the quasi-isotropic case ($E_x = E_y$) $P_{45} < 50\%$.

The maximum stress concentration is at point B (x=0,y=r) on Figure 3. When $E_x < E_y$ the point to be considered is A (x=r, y=0) and we shall have the same results if interchange x and y. When $E_x = E_y$ and $P_{45} \geq 50\%$ a rotation of the loading on 45^0 will cause a bigger stress concentration and this case will coincide with some of the other combinations.

As a result 36 laminates with different material properties are generated.

A unit thickness of the laminate is assumed. (It is not a simplification because it is the ratio t_L/t_R that is important) The thickness of the reinforcing ring is varied on five steps: 0.5, 1, 2, 3 and 4 times the thickness of the laminate.

The isotropic Young's modulus of the reinforcing ring is chosen to be $E_R = \frac{1}{2}(E_{x,L} + E_{y,L})V$, where V varies on six steps: 0.2, 0.33, 0.5, 1.5, 3.5, 6.

The Poisson's ratio of the reinforcing ring is chosen to be 0.3. (Test problems has shown that its variation gives negligible influence on the stress concentration factor.)

As a result 990 problems are generated for a given type of composite material and outside radius of the reinforcing ring. Having two types of composite materials (carbon fiber and glass fiber) and four values of R (R/r=1.5;2.0;3.0;4.0) it gives a total number of 7920 FEM problems which are considered adequate to reveal some empirical relationships.

Several test problems for infinite reinforcing ring (all elements defined twice) have been solved to check the procedures and the FEM meshes with the theoretical results given by eqn (25).

At the outer radius of the reinforcing ring the taper is neglected in the FEM model. This causes a stress singularity due to the abrupt change of thickness. A test problem has been run with an additional ring of tapered thickness elements but the influence on the stress concentration at the hole edge was negligible.

4. Processing of the Results

4.1. Establishing an empirical relationship

Since it has been assured that the discretisation error is small we shall regard the FEM results for the stress concentration factors $K_{t,L}^{FEM}$ and $K_{t,R}^{FEM}$, respectively, in the laminate and the reinforcing ring, when it is with finite dimensions, as the true ones.

The true correction factor Y^{FEM}, which is the target value for the estimated one Y^{E} defined by Eqn (28), is:

$$Y^{FEM} = \frac{K_{t,L}^{FEM}}{K_{t,L}^{\infty}} \qquad (29)$$

From closer inspection of the results it is observed that a certain pattern in the variation of Y^{FEM} can be achieved if it is considered as a function of X where:

$$X = \frac{E_R t_R}{E_{\theta,L} t_L} \qquad (30)$$

$E_R t_R$ is the isotropic stiffness per unit length and width of the reinforcing ring.

$E_{\theta,L} t_L$ is the stiffness per unit length and width of the laminate in direction of the stress σ_θ.

To establish the empirical relationship only the stress at point B (Figure 3, $\theta=90^{o}$, $E_{\theta,L} \equiv E_{x,L}$) will be considered. However this will not restrict the application of the relationship for other values of θ, as it is shown later in Section 6.

The values of Y^{FEM} are shown in Figure 8 as dots.

The solid line represent the estimated correction factor Y^{E}, to be computed by eqn (31) and to be used in eqn (28).

$$Y^{E} = aX^{b} \qquad (31)$$

The coefficients "a" and "b" are determined as follows:

- a geometric least squares fits are done for the carbon-fiber laminates for each case of R/r (R/r = 1.5, 2.0, 3.0 and 4.0.)
- a regression analysis of the four cases R/r is performed.

The expressions for "a" and "b" are:

$$a = 2.46\sqrt{r/R} \qquad \text{(if } a<1 \text{ then } a=1)$$

$$b = 0.40 + 0.22\ln(r/R) \qquad \text{(if } b<0 \text{ then } b=0) \qquad (32)$$

It is also assumed that if $R/r \geq 6$ the reinforcing ring may be treated as having infinite dimensions and a=1, b=0.

The functions $Y^{E} = aX^{b}$ for the considered ratio R/r in the FEM analyses are shown in Figure 9. The plot may be used for getting a quick, rough estimate for the correction factor Y^{E}.

For the glass-fiber laminates, which are more likely to be used in large scale constructions, there is an additional advantage that the empirical prediction of Y^{E}, eqns (31) and (32), is like an upper bound of the experimental numerical data and is on the safer side.

The graphs in Figure 8 concern the stress concentration in the laminate.

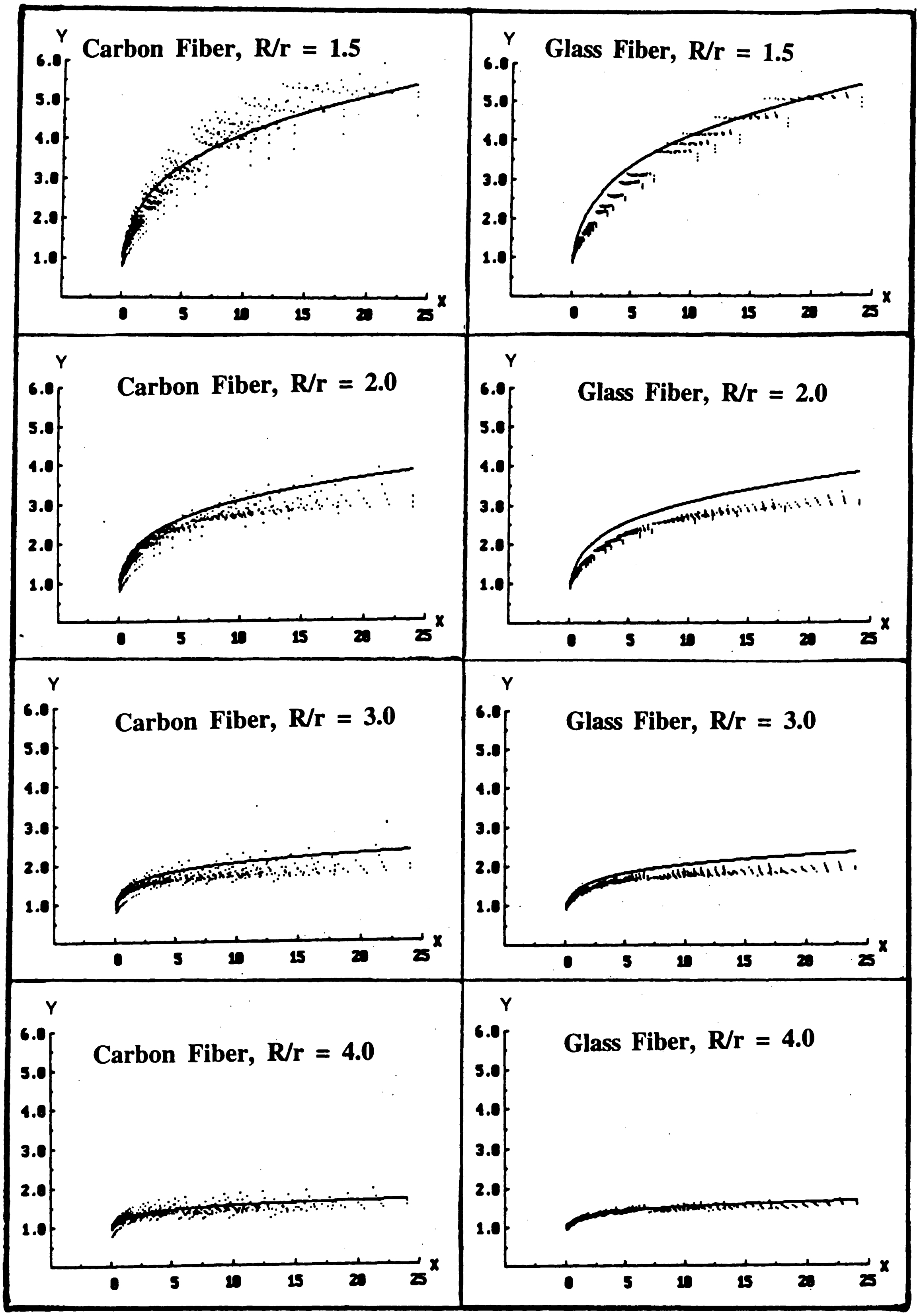

Figure 8.

Correction coefficient Y for the stress concentration in the laminate

Dots - $Y^{FEM} = K^{FEM}_{t,L} / K^{\infty}_{t,L}$; experimental, numerical data (990 dots)

Solid curve - $Y^{E} = aX^{b}$; empirical approximation

Left - Carbon fiber laminate; Right - Glass fiber laminate

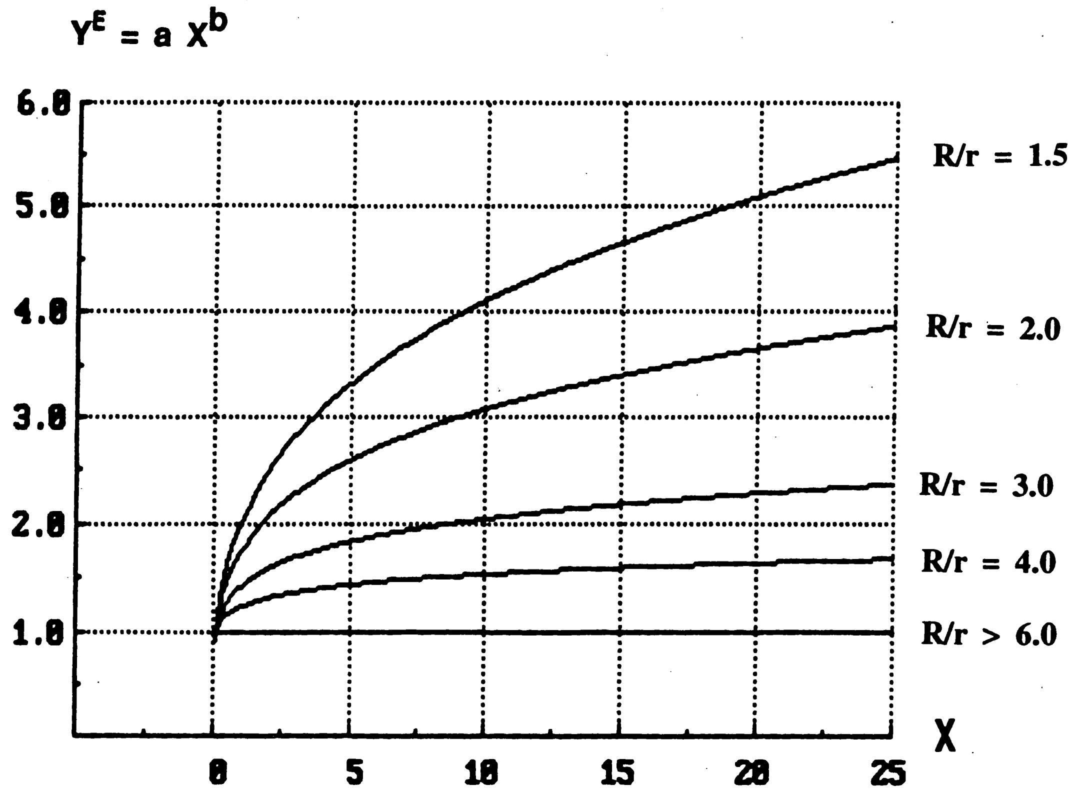

Figure 9.

A graph for a quick, rough estimation of Y^E

R/r = 1.5: a=2.00, b=0.310 R/r = 2.0: a=1.74, b=0.248

R/r = 3.0: a=1.42, b=0.158 R/r = 4.0: a=1.23, b=0.095

R/r > 6.0: a=1.00, b=0.000

If we consider the stress concentration in the reinforcing ring (eqn (29), subscript L to be changed to R) and plot the results, we shall get almost identical graphs. That is why they are not presented but the closeness can be observed in Table 3.

4.2. Accuracy of the empirical relationship

The Error of the estimated stress concentration factor in the laminate and the reinforcing ring is:

$$(\mathrm{Error})_{L,i} = \frac{K^{E}_{t,L} - K^{FEM}_{t}}{K^{FEM}_{t}} = \frac{Y^{E}K^{\infty}_{t,L} - K^{FEM}_{t}}{K^{FEM}_{t}} \qquad (\ i=1,N;\quad N=990\ )$$

$$(33)$$

$$(\mathrm{Error})_{R,i} = \frac{K^{E}_{t,R} - K^{FEM}_{t}}{K^{FEM}_{t}} = \frac{Y^{E}K^{\infty}_{t,R} - K^{FEM}_{t}}{K^{FEM}_{t}} \qquad (\ i=1,N;\quad N=990\ )$$

High values of the positive (maximum) error are more acceptable than high values of the negative (minimum) error because the former overestimates the stress concentration and is on the safer side.

As an overall measure of the accuracy one may consider:

$$\text{Mean Square Error} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\mathrm{Error}\right)_i^2} \qquad (34)$$

Subscripts L and R are omitted for simplicity, but the calculation is done for both cases.

The errors are given in Table 3.

	ERROR of the estimated stress concentration factor [%]											
	in the LAMINATE						in the REINFORCEMENT					
	Carbon-fiber laminate			Glass-fiber laminate			Carbon-fiber laminate			Glass-fiber laminate		
R/r	max.	min.	mean sq.	max.	min.	mean sq.	max.	min.	mean sq.	max.	min.	mean sq.
1.5	52	-26	15	41	-19	20	52	-27	15	41	-19	20
2.0	43	-20	12	31	-16	16	42	-22	12	30	-16	16
3.0	38	-14	11	25	-10	11	37	-15	10	24	-12	10
4.0	28	-17	7	13	-7	3	27	-16	6	12	-8	3

Table 3

Maximum, Minimum and Mean square error for $K^E_{t,L}$ and $K^E_{t,R}$

in a group of 990 test problems

The accuracy of the prediction of the stress concentration factor is regarded as acceptable. Looking for a better (and more complicated) expression for the correction factor Y^E is considered unnecessary because of the arbitrariness of the real load.

5. Numerical example

The stress distribution σ_θ around the edge of the hole, based on the procedures outlined so far, will be given for a test problem. When results are based on regression analysis they may be better for one problem than for another. To have an impartial choice, the problem of stress concentration under biaxial loading discussed in Reference [3] will be considered.

The laminate is $[(\pm45/0_2)_3]_S$ T800/924C. The material properties are:

E_x = 91.3 GPa; E_y = 26.1 GPa; G_{xy} = 23.6 GPa; ν_{xy} = 0.69

For the case of tension-compression loading a comparison of the results for points A and B as shown in Figure 3, is given in Table 4. Reference [3] uses the NISA FEM program, 200 8-node plane stress elements. In the present analysis the same elements are used and the mesh on Figure 7 has 130 elements. Better results with less elements are achieved due to the better grading of the mesh.

A unit tension (along direction of E_x) and a unit compression load (along direction of E_y) is applied. The numerical values of σ_θ are equal to the values for the stress concentration K_t.

Point A			Point B		
Theory	FEM Ref. 3	FEM Fig. 7	Theory	FEM Ref. 3	FEM Fig. 7
-2.87	-2.94	-2.906	5.37	5.41	5.397

Table 4

Stress concentration factors - Theory and FEM results

The comparison in Table 4 shows that the FEM results are quite reliable.

The stress σ_θ (and the numerically equal stress concentration factor K_t) in the case of infinite, soft reinforcing ring ($E_R \approx 0.5E_x$, $t_R/t_L=3$) is presented on Figure 10: graphs (1A) and (1B) show the distribution in the laminate, graphs (2A) and (2B) - in the reinforcing ring. There is a very close agreement between the theoretical distribution, eqn (22), graphs (1A),(2A), and the FEM distribution, graphs (1B),(2B).

For the case of finite reinforcing ring (R/r=4), the results are shown on Figure 11 for soft ring and Figure 12 for stiff ring, $E_R \approx 2E_x$. The graphs (...A) are the target distribution, graphs (...B) are the estimated distribution according to eqn (28).

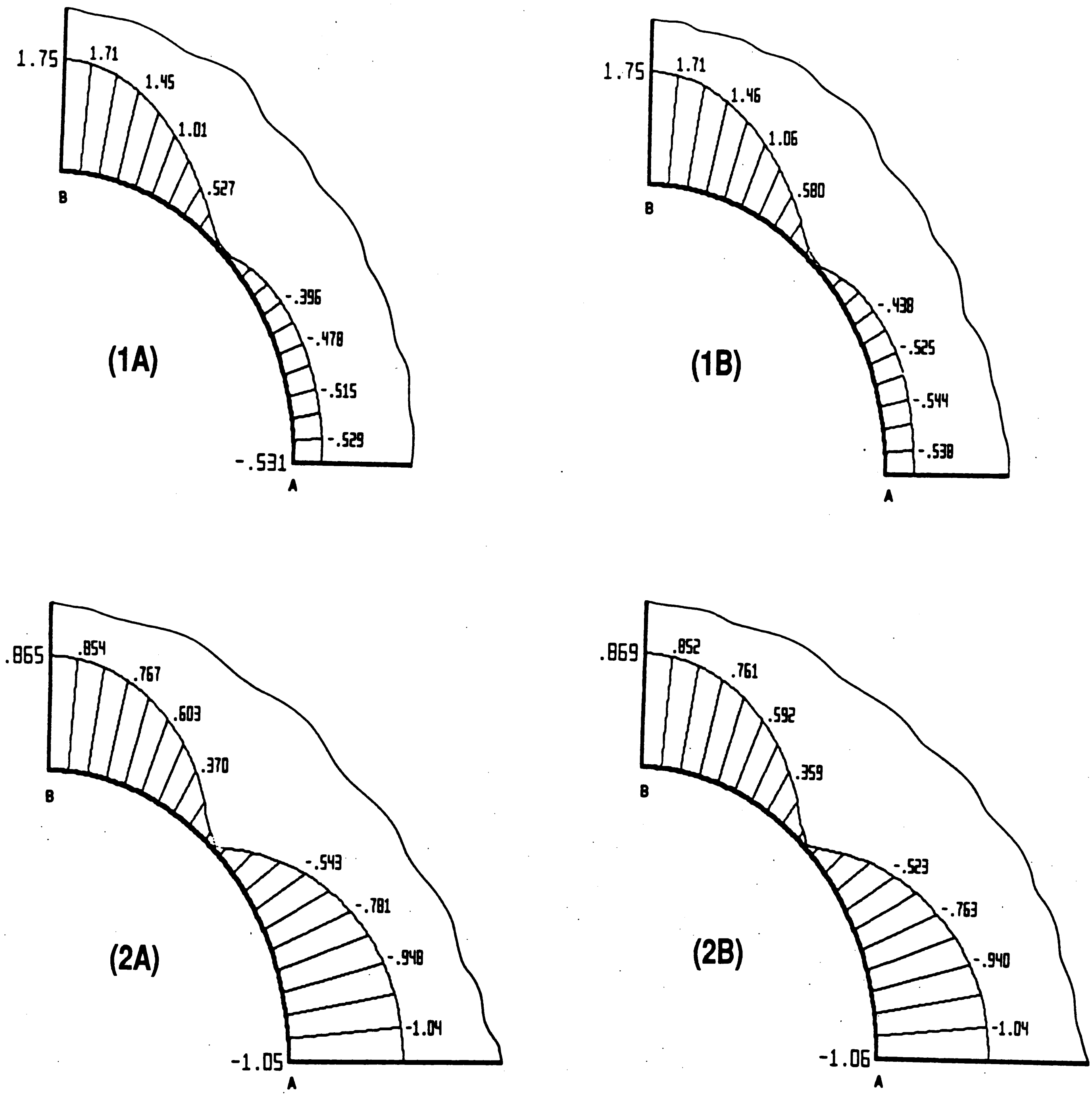

Figure 10

Distribution of σ_θ at the hole edge

(1A) - **Analytical** σ_θ in the **laminate** ; **infinite**, **soft** ring

(1B) - **FEM** σ_θ in the **laminate** ; **infinite**, **soft** ring

(2A) - **Analytical** σ_θ in the **reinf. ring** ; **infinite**, **soft** ring

(2B) - **FEM** σ_θ in the **reinf. ring** ; **infinite**, **soft** ring

(1A) and (2A) are the target distributions for (1B) and (2B), resp.

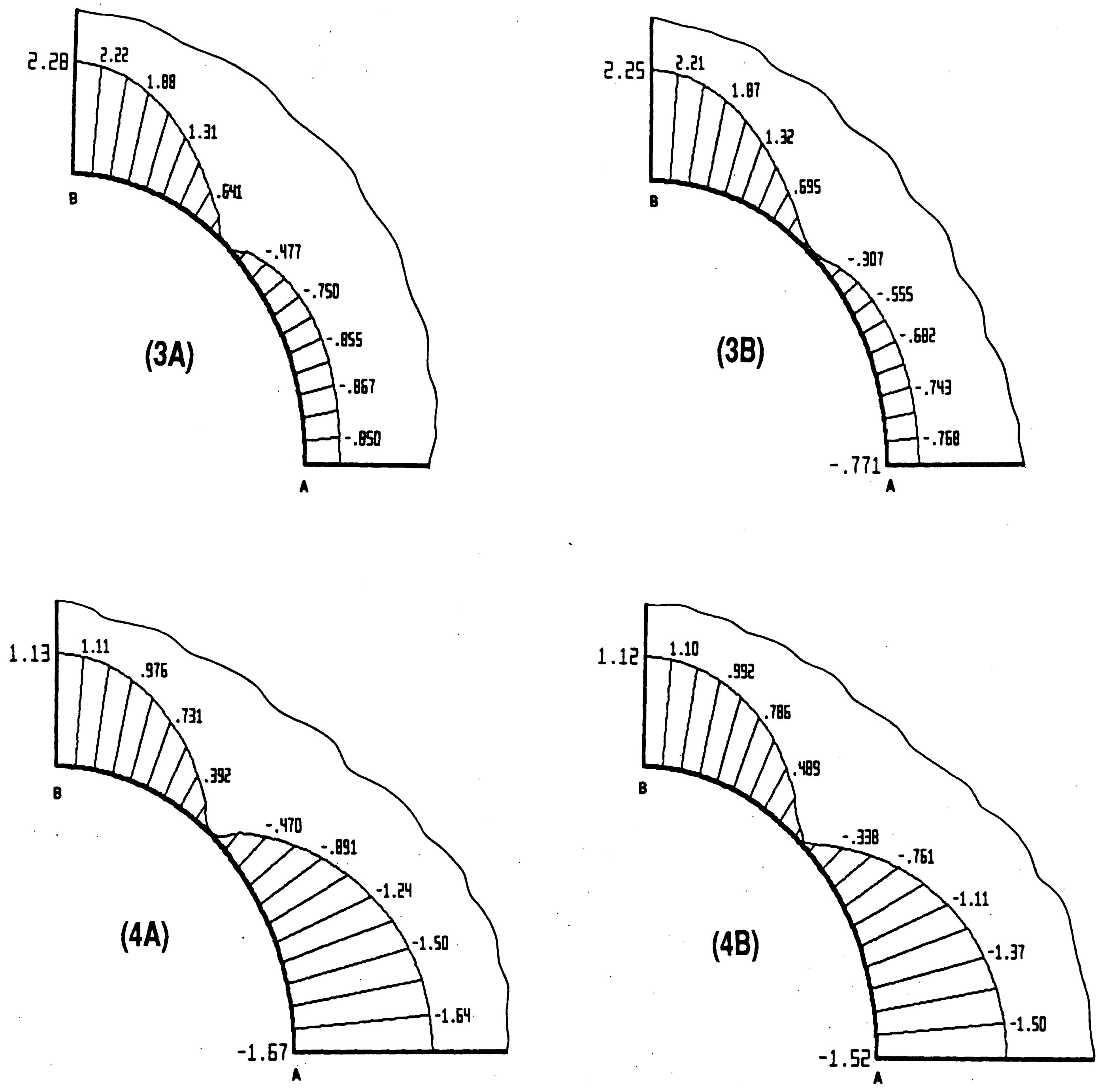

Figure 11

Distribution of σ_θ at the hole edge

(3A) - **FEM** σ_θ in the **laminate** ; **finite**, **soft** ring

(3B) - **Estimated** σ_θ in the **laminate** ; **finite**, **soft** ring

(4A) - **FEM** σ_θ in the **reinf. ring** ; **finite**, **soft** ring

(4B) - **Estimated** σ_θ in the **reinf. ring** ; **finite**, **soft** ring

(3A) and (4A) are the target distributions for (3B) and (4B), resp.

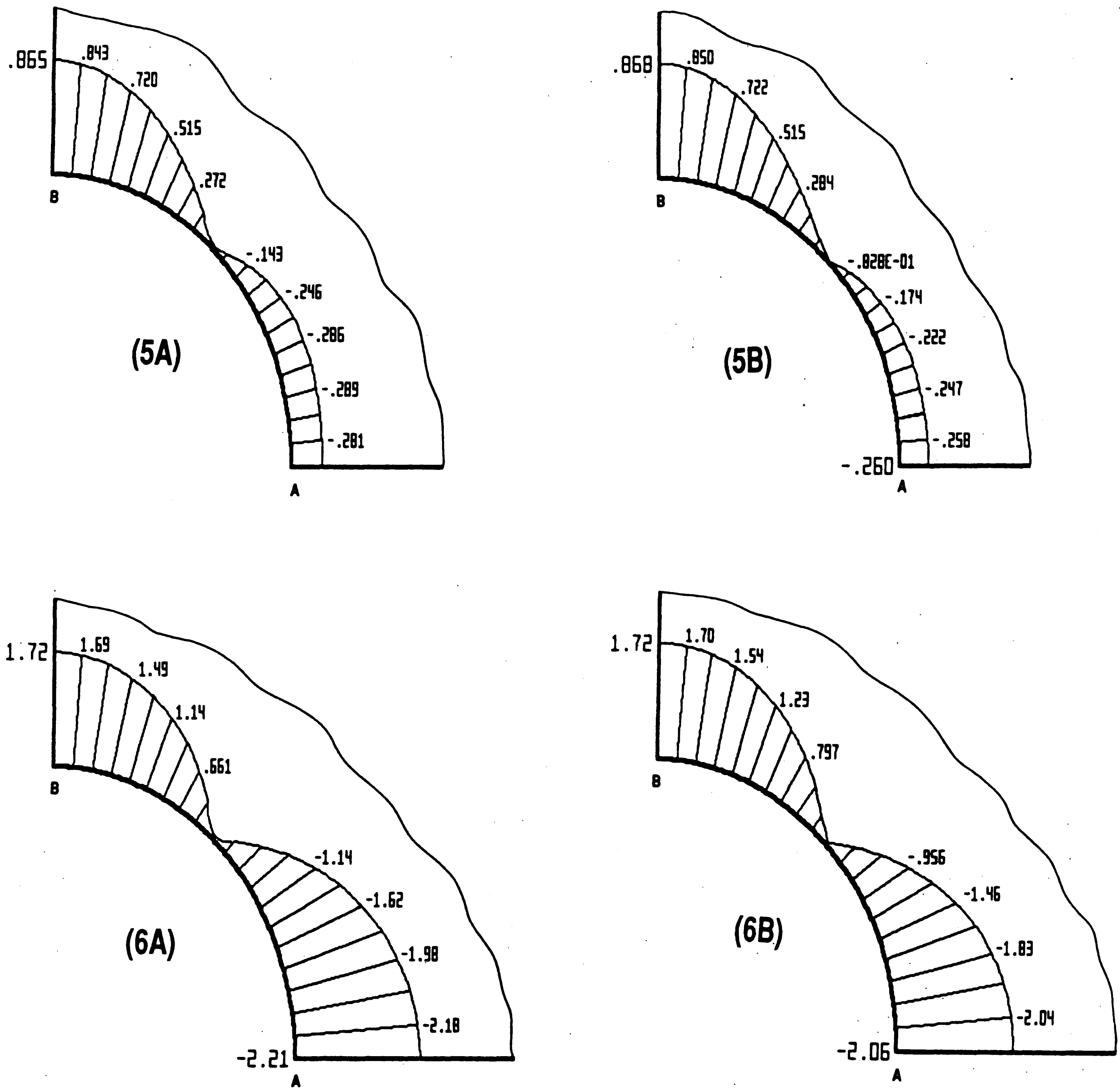

Figure 12

Distribution of σ_θ at the hole edge

(5A) - **FEM** σ_θ in the **laminate** ; **finite, stiff** ring

(5B) - **Estimated** σ_θ in the **laminate** ; **finite, stiff** ring

(6A) - **FEM** σ_θ in the **reinf. ring** ; **finite, stiff** ring

(6B) - **Estimated** σ_θ in the **reinf. ring** ; **finite, stiff** ring

(5A) and (6A) are the target distributions for (5B) and (6B), resp.

For the soft reinforcing ring the value of the correction factor, eqn (31), varies from Y^E=1.29 for point B to Y^E=1.45 for point A. The application of Y^E transforms graph (1A) to (3B) and (2A) to (4B). For the stiff reinforcing ring we have Y^E=1.47 and Y^E=1.65 respectively.

The correction factor Y^E, eqn (31), has been derived considering the stresses only for point B. At this point there is an excellent agreement between the estimated results shown in graphs (3B) to (6B) in Figures 11 and 12, with the FEM results, graphs (3A) to (6A). For point A the maximum error is less then 10% and the correction factor Y^E can be used for estimating the stress and the stress concentration factor anywhere on the edge of the hole.

Results for σ_θ (and the stress concentration factor for a unit load) in the case of unidirectional loading, when it is applied at an angle φ from the principal material axis x are presented in Table 5. For computation of K_t instead of eqn (14), eqn (1) is used. Point A is on the diagonal parallel to the loading, i.e. $\theta=\varphi$; point B is on the diagonal perpendicular to the loading, $\theta=\varphi+90^\circ$. The correction factor Y^E varies from 1.29 to 1.45.

			$\varphi = 0^\circ$	$\varphi = 15^\circ$	$\varphi = 30^\circ$	$\varphi = 45^\circ$	$\varphi = 60^\circ$	$\varphi = 75^\circ$	$\varphi = 90^\circ$
LAMINATE	point B	FEM	1.62	1.53	1.28	0.96	0.69	0.54	0.50
		Est.	1.64	1.57	1.36	1.06	0.79	0.64	0.59
	point A	FEM	-0.19	-0.21	-0.27	-0.38	-0.53	-0.66	-0.70
		Est.	-0.18	-0.19	-0.25	-0.35	-0.47	-0.57	-0.61
REINFORC.	point B	FEM	0.80	0.83	0.88	0.94	0.97	0.98	0.98
		Est.	0.81	0.84	0.93	1.03	1.12	1.16	1.17
	point A	FEM	-0.37	-0.37	-0.38	-0.38	-0.37	-0.36	-0.35
		Est.	-0.35	-0.35	-0.35	-0.34	-0.32	-0.31	-0.30

Table 5

FEM and Estimated σ_θ for a unit load at angle φ

There is a good agreement of the target, FEM, values and the estimated ones. This shows that the correction factor Y^E and the presented procedure for obtaining an estimated value for the stress concentration, although derived for a fixed loading (biaxial tension-compression), can be used when the loading is arbitrary.

6. Completeness of the analysis

For completeness of the analysis the tendency for delamination at the hole edge due to the through thickness peeling stress has to be taken into account[4,5].

If the assumption for infinite laminate is not valid appropriate provisions must be made. It is usually done by introducing a finite width correction factor[6]. If the geometry around the hole is more complex than the one used to derive this factor (which is most often the case) a FEM analysis is highly recommended as a final check of the results.

7. Conclusion - A Summary of the Procedure

When a hole is drilled in a laminated composite the stress concentration may be reduced by increasing the thickness of the laminate around the hole. The simplest way to increase the thickness is to bond two ring-shaped, isotropic panels (reinforcing ring) symmetrically on both sides of the laminate.

If the loading is varying, the biaxial tension-compression along the principal material axes of the laminate is to be considered as the most unfavorable one.

The maximum stress concentration factor at the edge of the hole can be determined according to the following procedure:

P1. Compute the compliance matrix [S] for the assembly laminate and reinforcing ring. (The subscript A, used previously for all terms with reference to the assembly, is dropped out for simplicity.)

$$[S] = \left(t_L + t_R\right) \left[[Q]_L t_L + [Q]_R t_R\right]^{-1} \qquad (\text{P1})$$

P2. Recover the material properties E_x, E_y, G_{xy}, ν_{xy} for the assembly from its compliance matrix [S].

$$E_x = \frac{1}{S_{11}} \qquad E_y = \frac{1}{S_{22}}$$

$$G_{xy} = \frac{1}{S_{66}} \qquad \nu_{xy} = - E_x S_{12} \qquad (\text{P2})$$

P3. Using the material properties for the assembly compute the stress concentration factor for the assembly (infinite reinforcing ring; biaxial tension-compresion):

$$K_t^{\infty} = (1 + n + k) \qquad (\text{P3})$$

where:

$$k = \sqrt{\frac{E_x}{E_y}}$$

$$n = \sqrt{2\left(k - \nu_{xy}\right) + \frac{E_x}{G_{xy}}}$$

Remark:

If the allowable remote stress for the unnotched laminate is different for the two principal material directions then eqns (12) and (13) must be used.

If the loading is not along the principal material axes or an estimate of the stress concentration at another point is required then eqn (1) must be used.

P4. Compute coefficients C_L and C_R which show how the stress σ_θ in the assembly, around the edge of the hole, is distributed in the laminate and the reinforcing ring ($E_{x,L}>E_{y,L}$).

$$C_L = \frac{E_{x,L}t_L}{E_{x,L}t_L + \xi E_R t_R}$$

$$C_R = \frac{E_R\ t_L}{\frac{1}{\xi}E_{x,L}t_L + E_R t_R} \qquad (P4)$$

where:

$$\xi = \frac{E_{y,L}t_L\left(1 - v_{xy,L}v_R\right) + E_R t_R\left(1 - v_{xy,L}v_{yx,L}\right)}{E_{y,L}t_L\left(1 - v_R\ v_R\right) + E_R t_R\left(1 - v_R\ v_{yx,L}\right)}$$

P5. Compute the correction coefficient Y^E:

$$Y^E = aX^b \qquad (P5)$$

where:

$$X = \frac{E_R t_R}{E_{x,L}t_L}$$

$$a = 2.46\sqrt{r/R} \qquad \text{(if a<1 then a=1)}$$

$$b = 0.40 + 0.22\ln(r/R) \qquad \text{(if b<0 then b=0)}$$

P6. The estimated stress concentration factor in the laminate and the reinforcing ring, respectively, is:

$$K^E_{t,L} = Y^E\ C_L K^\infty_t$$

$$K^E_{t,R} = Y^E\ C_R K^\infty_t \qquad (P7)$$

By iterative calculations, say, constant material properties of the reinforcing ring and varying thickness, one may compensate fully ($K_t^{\infty}=1$) the hole-induced stress concentration.

The accuracy of the estimated stress concentration factor is considered very good. In most cases the uncertainty of the load will be much greater and the exaggeration of having biaxial tension-compression loading will compensate for any underestimating of the stress concentration.

Acknowledgement

The work reported here was performed at Oslo University, Department of Mathematics. It has been made possible thanks to one year Post-Doctoral research fellowship grant ST.60.61.222052 given by the Norwegian Research Council to Dr. Rosen Tenchev.

References

1. Lekhnitskii, S.G., *Anisotropic Plates,* Gordon & Breach, New York, 1968.

2. Powell, P.C., *Engineering with Polymers,* Chapman & Hall

3. Pateau, V. & Soutis, C., *Multiaxial loading of composite laminates,* Report 91-32 Dec. 1991, Leicester University, UK.

4. Hart-Smith, L.,J., *Designing to minimize peel stresses in adhesive-bonded joints,* Paper 7389, ASTM Symp. on Delamination and Debonding of Materials, Pittsburgh, Pennsylvania, Nov. 8-10, 1983.

5. Shenoi, R.A. & Wellicome, J.F. (eds), *Composite Materials in Maritime Structures,* Vol. 1., Chapter 7.7, Cambridge University Press, 1993.

6. Tan, S.C., *Finite-Width Correction Factors for anisotropic Plate Containing a Central Opening,* J. Composite Materials, pp. 1080-1097, Vol. 22, Nov. 1988.

APPENDIX A

MATERIAL STIFFNESS MATRIX of a LAMINATED COMPOSITE

The material stiffness matrix of a symmetric composite laminate along axes x,y is:

$$[Q] = \frac{1}{t_c} \sum_{i=1}^{n} [Q_{xy}]_i \, t_i$$

where:

n - number of plies

t_c - thickness of the laminate

t_i - thickness of ply i

$[Q_{xy}]_i$ - the material stiffness matrix of ply i along x,y

$$[Q_{xy}]_i = [T]_i^T \, [Q_{12}]_i \, [T]_i$$

$[Q_{12}]_i$ - the material stiffness matrix of ply i along its principal material directions 1 and 2.

(1 - along the fibers, 2 - transverse to the fibers)

$[T]_i$ - transformation matrix

$$[T]_i = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix}$$

θ - angle from axis x to axis 1 of ply i